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Modeling and Inference for Transient Tracers on Oceanic Flow
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Executive Summary

This report surveys recent progress in research on modeling and inference for transient tracers on oceanic flows. The survey covers in some detail the contents of five technical reports on various aspects of this research, including a Markov process for mass transport and estimation functions for estimating flow coefficients, dissipation rates, and birth rates from chronicles of transient tracers. The details include theoretical results of independent interest that support applications to inference from tracer data.

A) Modeling Mass Transport. Our approach considers stochastic descriptions of mass transport by fluid flows. The particular model is one of birth and death on a Brownian flow, describing the injection, transport, and dissipation of tracer elements on a turbulent fluid. As treated by Çinlar and Kao, the mean behavior of this particle system is governed by an advection-dispersion equation of some interest to scientists studying transient-tracer problems. We show in Phelan [3] that this system induces a Feller Markov process on the space of counting measures and we calculate its generator on a subdomain.

B) Dynamics of Mass Transport. The dynamics of mass transport here refers to the martingale dynamics of the particle process tracking the system of particles on the flow. In Phelan [1] and [4], we build on the treatment in Çinlar and Kao and calculate a set of characteristics that inform our treatment of a stochastic integral on the particle process and of a martingale problem. The stochastic integral and the well-posedness of the martingale problem play a key role in our treatment of estimation from transient tracers on flows.

C) Statistical Inference. The martingale dynamics of mass transport are essential to our approach to estimation for birth and death on flows. They inform a class of estimation functions on the particle process including inference from chronicles on transient tracers in Phelan [1] and inference from integral data on the same in Phelan [2]. A related issue in Phelan [5] concerns the specification of a density process as needed for maximum-likelihood estimation. In this case, we get a class of optimal estimators for drift rates of the flow, injection rates of new mass, and dissipation rates of the old. This is the subject of our current working paper.

D) Continuing Research. Our continuing research moves in three broad directions. We are now in a position to further develop methodology for inference for models of mass transport on fluids, including parameter estimation, sampling issues, and state estimation as well. This in turn motivates a serious look at the computational issues arising from application of the methods to data drawn from fields experiments. Finally, we remain interested in studies of equilibrium-particle systems as well as in extensions that include chemical interactions.

Technical Reports in Support of this Research
Submitted for Publication

1. Phelan, M.J. (1992a) Estimation Functions on a Particle Process, University of Pennsylvania Technical Report, Philadelphia, PA.

2. Phelan, M.J. (1992b) A Quasi Likelihood for Integral Data on Birth and Death on Flows, University of Pennsylvania Technical Report, Philadelphia, PA.

3. Phelan, M.J. (1993a) On the Transition Semigroup of a Birth and Death on a Flow, University of Pennsylvania Technical Report, Philadelphia, PA.

4. Phelan, M.J. (1993b) A Markov Process and A Martingale Problem, University of Pennsylvania Technical Report, Philadelphia, PA.

5. Phelan M.J. (1993c) A Density Process on a Particle Process, University of Pennsylvania Technical Report, Philadelphia, PA.

1. INTRODUCTION

We consider a particle system of birth and death on a Brownian flow. This refers to Çinlar and Kao's 1992 model of mass transport by a fluid flow. They imagine tracer elements on a turbulent fluid. A Brownian Flow describes the fluid-flow map of this fluid over its domain. A Poisson Process regulates the birth of particles that live and die there. Like tracer elements, their motion yields to the motion of the flow. And they dissipate in response to position-dependent killing or decay. Eventually, they die and leave the flow.

This particle system induces a particle process. At each time, this process tells the configuration of existing particles on the domain of the flow. It is a Markov process on the space of point measures on the same.

Çinlar and Kao (1992a&b) study the particle process in sundry ways. They analyze its limiting behavior as time approaches infinity. And they give its martingale dynamics and distributional descriptors. The latter include partial differential equations for the mean, covariance, and Laplace transform.

We survey some work in Phelan (1992,1993). In particular, we identify a set of local characteristics for the particle process. These provide a basis for writing a stochastic integral. We treat the transition semigroup of the process, showing that it is a Feller semigroup and exhibiting its generator explicitly on a subset of functionals in its domain. This provides the basis of a martingale problem having the particle process as its unique solution. Along the way, we discuss the utility of these results in problems of statistical inference.

2. A MARKOV PROCESS

We describe a birth and death on a stochastic flow. Our description begins with a Brownian flow on Euclidean space and then introduces the particle system. The probability space $(\Omega, \mathcal{H}, \mathbb{P})$ supports all of the random variables appearing below.

Stochastic Flows

The set E denotes Euclidean space of dimension d. A Stochastic Flow $F = (F_{st}), 0 \leq s \leq t \leq \infty$ on E is a family of transformations of E satisfying:

$$(1) \qquad F_{ss} = I \text{ and } F_{tu} \circ F_{st} = F_{su}, \; 0 \leq s \leq t \leq u.$$

As usual, I denotes the identity transformation and $\circ$ denotes the composition. For x in E, $s \geq 0$, and ω in Ω, let us visualize a particle sitting at position x at time s. If its position in E at time t is $F_{st}^{\omega}x$, then the mapping $t \to F_{st}^{\omega}x$ is the particle's trajectory on the flow. This motion is a one-point flow on F.

We assume that F is a Brownian Flow. Brownian Flows may arise as solutions to stochastic differential equations. For example, let W be a Weiner process on m-dimensional Euclidean space. Also, let γ be a mapping from E to the space of $d \times m$ matrices, and let b be a mapping from E to E. The coefficients b and γ satisfy the global Lypschitz condition:

$$|b(x) - b(y)| + \|\gamma(x) - \gamma(y)\| \leq K|x - y|, \tag{2}$$

and the linear growth condition,

$$|b(x)| + \|\gamma(x)\| \leq K(1 + |x|), \tag{3}$$

for some constant K.

We assume that $t \to F_{st}x$ is the solution of Itô's differential equation:

$$dX_t = b(X_t)dt + \gamma(X_t)W(dt), \quad X_s = x, \tag{4}$$

for every $t \geq s$ and $x \in E$. This obtains a unique Brownian flow of homeomorphisms. We refer to Kunita(1990) for details.

In our case, the one-point motions are diffusions having infinitesimal mean b and infinitesimal covariance c satisfying: $c(x,y) = \gamma(x)\gamma^{\top}(y)$, for $x, y \in E$, the $\top$ for transpose. Thus, the Markov generator A of one-point motion satisfies the equation:

$$Af(x) = \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} c_{ij}(x,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x), \tag{5}$$

for every function having two continuous derivatives. In general, it takes two-point motions to determine a Brownian flow, since it takes two to determine the infinitesimal parameters b and c.

Birth and Death on Flows

The particle system is a countable collections of particles that enter, live, and die on the stochastic flow. A point process heralds the entry of these

particles, giving each an initial allowance of time on the flow. As this time expires, they die and leave the flow.

The generic particle enters the flow at random time S and random position X. It leaves the flow at time T, a positive time strictly after S. If the particle is alive at time t, $S \leq t < T$, then its position is $F_{St}X$. This motion stops when the particle dies and leaves the flow.

The cause of death lies in the cumulative effect of killing. This effect exhausts the particle's initial allowance U of time on the flow. The killing rate k is a positive Borel function on E. The time of death T then satisfies:

$$T = \inf\{t > S : \int_S^t dr k \circ F_{Sr}X > U\}. \tag{6}$$

This specifies the birth, life, and death of particles on the flow.

We label the system of particles in some way. The particle of label p inherits the endowment (S_p, X_p, U_p) at birth; the time T_p comes with death. The endowments form the atoms of a point process, the life process L of Çinlar and Kao (1992). The life process regulates birth and death on the flow, which itself regulates motion. The assumption below is that L is a Poisson Random Measure enjoying independence from the flow.

We track the configuration of live particles on the flow with the particle process. This is the measure-valued process $t \to M_t$ on the space of point measures on E. For each t and Borel subset B of E, $M_t(B)$ satisfies the equation:

$$M_t(B) = \sum_p 1_B(F_{S_pt}X_p)1(S_p \leq t < T_p), \tag{7}$$

thus counting the number of live particles in B at time t. The atoms of M_t then determine the configuration of living particles.

The hypotheses are as follows. The life-flow pair (L, F) drives the particle system. The flow F is a Brownian Flow having infinitesimal mean b and infinitesimal covariance c. The life process L is a Poisson Random Measure that is independent of the flow. Its mean measure λ satisfies:

$$\lambda(ds, dx, du) = \delta_0(s)\mu_0(dx)due^{-u} + (1 - \delta_0(s))ds\pi(dx)due^{-u}, \tag{8}$$

for every $s \geq 0, x \in E, u \geq 0$, where δ_0 is the Dirac measure at zero and μ_0 and π are finite measures on E. These hypotheses make a Markov process of the particle process.

3. CHARACTERISTICS

The particle process is a Markov process on the space of point measures on E. In this section, we calculate its characteristics. In this direction, we start with a result from Çinlar and Kao (1992a).

A Semimartingale

The stochastic basis is $(\Omega, \mathcal{H}, \mathbf{H}, I\!P)$. The filtration $\mathbf{H} = (\mathcal{H}_t), t \geq 0$, satisfies:

$$\mathcal{H}_t = \sigma(L([0,s] \times B), B \in \mathcal{E} \otimes \Re_+, F_{sr}, 0 \leq s \leq r \leq t), \tag{9}$$

where $\mathcal{E} \otimes \Re_+$ denotes the Borel subsets of $E \times I\!R_+$.

The birth process N is the restriction of L to $(0,\infty) \times E$, so its atoms are the (S_p, X_p) among those particles born after time zero. The atoms of the killing field K, in contrast, are the time and place of death, $(T_p, F_{S_p T_p} X_p)$, of all particles. Finally, the notation $M_t f$ denotes the integral of the function f relative to M_t. The function belongs to the space $\mathbb{C}_K$ of continuous functions having compact support in E or its subspace $\mathbb{C}_K^2$ of twice continuously differentiable functions.

1 Proposition. *For each f in $\mathbb{C}_K^2$, the process $t \to M_t f$ is a semimartingale satisfying the stochastic differential equation:*

$$dM_t f = N(dt, f) - K(dt, f) + M_t(Af)dt + \sum_{j=1}^{m} M_t(((\nabla^\top f)\gamma)_j) W^j(dt),$$

where A satisfies Equation 5 and $\nabla^\top = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$. Also, $N(dt, f)$ is the integral of $N(dt, dx) f(x)$ over x in E, with $K(dt, f)$ the same of K and f.□

Çinlar and Kao (1992a) prove this proposition in their Lemma 3.2. They use the fact that the particle motions are themselves semimartingales and an application of Itô's formula. We use it to write the characteristics of the particle process.

Characteristics

The characteristics of the particle process are analogues of the triplet of local characteristics of ordinary semimartingales. They extend the natural ideas of trend, quadratic variation, and compensation of jumps to this measure-valued process.

First, we prepare notation. A Radon measure on E is a measure on $(E, \mathcal{E})$ that is finite on all bounded sets in $\mathcal{E}$. A signed-Radon measure on E is the difference between two Radon measures on E. The spaces $(M_b, \mathcal{M}_b)$ and $(M, \mathcal{M})$ denote the respective spaces of bounded, counting measures and signed-Radon measures on E with their Borel sigma algebra relative to the vague topology. The former is the state space of the particle process.

For ξ in M, let ξf be the integral of the measurable function f relative to ξ. In contrast, let $f\xi$ be the signed Radon measure having density f relative to ξ. In this way, the integral of the measurable function g relative to $f\xi$ may appear as $(f\xi)g$ or $\xi(fg)$.

Let A again be the Markov generator of one-point motion on the flow, satisfying Equation 5 . Let D be the differential operator satisfying the equation:

$$Dh(x,y) = \sum_{i=1}^{d}\sum_{j=1}^{d} c_{ij}(x,y)\frac{\partial^2 h}{\partial x_i \partial y_j}(x,y), \tag{10}$$

for every function on $E \times E$ having two continuous derivatives. Its domain includes the tensor product $f \otimes g$, namely $(x,y) \to f(x)g(y)$, for every pair of functions in $\mathbb{C}^2_K$. Next, let β denote the mapping $(m,f) \to \pi f - (km)f + m(Af)$ on $M_b \times \mathbb{C}^2_K$. Let q denote the mapping $(m,f,g) \to (m \times m)(D(f \otimes g))$ on $M_b \times \mathbb{C}^2_K \times \mathbb{C}^2_K$, where $m \times m$ is the product measure of m with itself. Finally, let κ denote a transition kernel from $(M_b, \mathcal{M}_b)$ into $(M, \mathcal{M})$. For each m, it charges only the signed-Dirac measures on E and satisfies the equation:

$$\kappa(m; d\eta) = \pi \circ \delta^{-1}(d\eta) 1_{\{+1\}}(\eta 1) + (km) \circ \delta^{-1}(-d\eta) 1_{\{-1\}}(\eta 1), \tag{11}$$

for every signed-Dirac measure η in M, where δ is the mapping from E to the space of Dirac measures on E.

The characteristics of the particle process refer to a triplet (B, C, ν) on the stochastic basis $(\Omega, \mathcal{H}, \mathbf{H}, \mathbb{P})$. For each f and g in $\mathbb{C}^2_K$, it satisfies:

$$\begin{aligned} dB_t(f) &= dt\beta(M_{t-}; f) \\ dC_t(f \otimes g) &= dtq(M_{t-}; f, g) \\ \nu(dt, d\eta) &= dt\kappa(M_{t-}; d\eta), \end{aligned} \tag{12}$$

where M_{t-} is the vague limit of M_s as s increases to t. This next proposition motivates characteristics as this triplet's name.

2 Proposition. *For each f in $\mathbb{C}_K^2$, let T_f denote the mapping $(t, \eta) \to (t, \eta f)$ on the product space $\mathbb{R}_+ \times M$. The triplet $(B(f), C(f \otimes f), \nu \circ T_f^{-1})$ then assembles the local characteristics of the semimartingale $t \to M_t f$.*

For this same semimartingale, $C(f \otimes g)$ is its second characteristic of quadratic covariation with the semimartingale $t \to M_t g$, for every g in $\mathbb{C}_K^2$.

This proposition is an easy consequence of Propostion 1 and classical theory of semimartingales and stochastic integration. Phelan (1992a) treats this result in detail. We use the characteristics next to develop a stochastic integral on the particle process and below to exhibit the generator of the particle process on a subset of its domain.

Stochastic Integral

We introduce a stochastic integral on the particle process here. The integral is a special one covering our construction of estimation functions in Phelan (1992a). Itô (1984) studies stochastic integrals for processes on the space of distributions. We adopt his framework and view the particle process in this way.

The space $\mathcal{D}$ is the space of infinitely differentiable functions having compact support in E. The space $\mathcal{D}'$ denotes the dual to $\mathcal{D}$ or the space of distributions on E. We endow $\mathcal{D}'$ with the Schwartz topology, which is the usual topology for this space; cf. Conway(1985)[p120].

We introduce a stochastic differential equation for the particle process as a process on $\mathcal{D}'$. In particular, the particle process admits the decomposition given by the following equation:

$$
\begin{aligned}
M_t &= M_0 + B_t + X_t \\
B_t &= \int_0^t ds[\pi + kM_{s-} + A^* M_{s-}], \quad t \geq 0,
\end{aligned}
\tag{13}
$$

implicitly defining the measure-valued martingale $t \to X_t$, where A^* denotes the adjoint of A as an operator on distrubutions. We further decompose the martingale part below.

We identify a stochastic integral on the martingale part of the particle process here, beginning as usual with simple integrands. A simple integrand

is a process $t \rightarrow H_t$ of the form: $t \rightarrow Y1_{(r,s]}(t)$, where $s > r$ and Y is a linear functional on $\mathcal{D}'$. Moreover, for each distribution T in $\mathcal{D}'$, the action YT of Y on T is $1_G T\phi$ where G is a measurable set in $\mathcal{H}_r$ and ϕ is a function in $\mathcal{D}$. The stochastic integral $H \cdot X$ of H with respect to X then satisfies:

$$H \cdot X_t = Y[X_{s\wedge t} - X_{r\wedge t}] = 1_G[X_{s\wedge t}\phi - X_{r\wedge t}\phi], \quad t \geq 0. \tag{14}$$

This integral defines a martingale on the particle process. We extend it in a natural way to a modest class of integrands.

Our class of integrands meets three criteria. First, we assume that an integrand $t \rightarrow H_t$ is a predictable mapping where the predictable sigma algebra is that induced by the predictable rectangles $\{(s,t] \times A \times B; s < t, A \in \mathcal{E}, B \in \mathcal{H}_s\}$ relative to the filtration $\mathbf{H} = (\mathcal{H}_t), t \geq 0$ of Equation 9. Second, we assume it take values in the space of linear functionals on $\mathcal{D}'$ by indentification with the predictable mapping $t \rightarrow h_t$ of functions in $\mathcal{D}$. That is, for each t, the functional H_t is the mapping $T \rightarrow Th_t$ on $\mathcal{D}'$. Third, we impose the following integrability condition:

$$\mathbb{E}\int_0^t dsq(M_{s-}; h_s, h_s) + \mathbb{E}\int_0^t \int_M \nu(ds, d\eta)(\eta h_s)^2 < \infty, \quad t > 0. \tag{15}$$

The operator q and the random measure ν belong to the characteristics of the particle process at Equation 12.

These criteria introduce a vector space Ψ of predictable mappings. For each $t \geq 0$, let p^t denote the seminorm on Ψ satisfying the equation:

$$p^t(H) = \{\mathbb{E}\int_0^t dsq(M_{s-}; h_s, h_s) + \mathbb{E}\int_0^t \int_M \nu(ds, d\eta)(\eta h_s)^2\}^{\frac{1}{2}}, \tag{16}$$

for every H in Ψ. The family $\{p^t, t \geq 0\}$ of seminorms induces a topology $\mathcal{T}$, making $(\Psi, \mathcal{T})$ a topological vector space. If the family is separating, then $(\Psi, \mathcal{T})$ is uniformly isomorphic to a dense subset of a complete topological vector space $(\hat{\Psi}, \hat{\mathcal{T}})$. We suppose that this is so and refer to the completed space as our class of X-integrable processes.

3 Proposition. *We suppose that the family $\{p^t, t \geq 0\}$ of seminorms on Ψ satisfies the equation: $\bigcap_{t\geq 0}\{H : p^t(H) = 0\} = \{0\}$. The mapping $H \rightarrow H \cdot X$ for H among the simple integrands then extends, using the same notatation, to a mapping $H \rightarrow H \cdot X$ for H among the integrands in our class of X-integrable processes. The resulting integral process $t \rightarrow H \cdot X_t$ gives a locally*

square-integrable martingale on the particle process. Its quadratic characteristic $t \to \langle H \cdot X \rangle_t$ satisfies the equation:

$$\langle H \cdot X \rangle_t = \int_0^t dsq(M_{s-}; h_s, h_s) + \int_0^t \int_M \nu(ds, d\eta)(\eta h_s)^2.$$

We apply this integral in Phelan (1992a) in defining estimation functions on the particle process. In doing so, we separate an integral on the ' discrete part' of X from one on its 'continuous part.' That is, the discrete part belongs of course to the compensated-point process $\mu - \nu$. The particle process then satisfies the representation of the following equation:

$$M_t = M_0 + B_t + X_t^c + \int_0^t \int_M (\mu - \nu)(ds, d\eta)\eta, \quad t \geq 0, \tag{17}$$

implicitly defining the continuous part $t \to X_t^c$. The terms in X^c and in $\mu - \nu$ determine respectively the continuous-martingale part and the discrete-martingale part of the the particle process.

Now, for suitable integrand H and random field Z, we may introduce the integral processes $t \to H \cdot X_t^c$ and $t \to Z * (\mu - \nu)_t$ on these two parts. The former is in accord with the arguments of Proposition 3, the latter in accord with standard theory of stochastic integration over point processes; cf. Definition II.1.27 [p72] of Jacod and Shiryaev (1987). This Z need not come from an H as in Proposition 3, but when it does come so the sum of these two integrals agrees with the integral of H with respect to $M - B$.

Corollary. *Let H and H' be X-integrable processes in Ψ. And let Z and Z' be $(\mu - \nu)$-integrable random fields in the class $G_{loc}(\mu)$ of Definition II.1.27 of Jacod and Shiryaev (1987). Finally, in accord with the same definition and Propostion 3, let Y and Y' denote the respective integral processes $t \to H \cdot X_t^c + Z * (\mu - \nu)_t$ and $t \to H' \cdot X_t^c + Z' * (\mu - \nu)_t$.*

These processes are locally square-integrable martingales. Their quadratic covariation $t \to \langle Y, Y' \rangle_t$ satisfies the equation:

$$\langle Y, Y' \rangle_t = \int_0^t dsq(M_{s-}; h_s, h'_s) + \int_0^t \int_M \nu(ds, d\eta) Z(s, \eta) Z'(s, \eta),$$

where $t \to h_t$ and $t \to h'_t$ are the respective $\mathcal{D}$-valued processes belonging to H and H'. □

Finally, we can develop such an integral on the particle process itself. In particular, let H be an element of Ψ with supporting process $t \to h_t$. Let Z denote the random field $(t,\eta) \to \eta h_t$ on $\mathbb{R}_+ \times M$ and let $t \to T_t$ denote the mapping $t \to \pi + kM_t + A^*M_t$. In this case, there is an integral process $t \to H \cdot M_t$ satisfying the equation:

$$(18) \quad H \cdot M_t = \int_0^t du T_u h_u + Z * (\mu - \nu)_t + \sum_{k=1}^{m} \int_0^t (\Gamma_k M_u)(h_u) W^k(du),$$

yielding a semimartingale of the particle process.

4. A TRANSITION SEMIGROUP AND ITS GENERATOR

We discuss the transition semigroup belonging to the particle process. Our aim is to establish that the transition semigroup is a Feller semigroup. We then exhibit the action of its generator on a subclass of functionals explicitly in terms of the characteristics. The latter informs the basis of our martingale problem.

Transition Semigroup

The particle process takes values in the space M_b of bounded, counting measures on E. We endow this space with the Borel sigma algebra $\mathcal{M}_b$ relative to the topology of vague convergence. The space $C_0(M_b)$ denotes the subspace of bounded, vaguely continuous functionals on M_b that vanish at infinity. It is a Banach space in the topology of uniform convergence. We let $\|\cdot\|_0$ denote the corresponding norm.

The state space is a locally compact Hausdorff space whose topology has a countable base. That is, let f_k be a sequence of functions in $\mathbb{C}_K$ that increase to 1 on E. For each bounded, counting measure m_0 and positive δ, the set $\cap_k\{m : mf_k \le m_0 f_k + \delta\}$ contains m_0 and is relatively compact in M_b with respect to the vague topology by virtue of A.15.7.5 of Kallenberg (1986). The vague toplogy has a countable base by virtue of A.15.7.7 of Kallenberg (1986) and Proposition 7.6 of Royden (1968).

We introduce the Markov transition semigroup $(P_t), t \ge 0$ belonging to the particle process. This next proposition shows that it is a Feller semigroup.

4 Proposition. *We suppose that the system parameter (b, c, π, k) satisfies the regularity conditions of Section 2. We suppose that the killing rate k is*

globally Lypschitz continuous and bounded on E. The transition semigroup $(P_t), t \geq 0$ is then a strongly continuous, contraction semigroup on $C_0(M_b)$.

The idea of the proof is to show that P_t maps $C_0(M_b)$ into itself for every t in $I\!R_+$ and to show that $P_t\varphi$ converges pointwise to φ as t descends to zero for every φ in $C_0(M_b)$. That $(P_t), t \geq 0$ is a Feller semigroup then comes of the argument of Exercise 9.27 in Sharpe (1988). The details appear in Phelan (1993a).

Generator

The generator G of the particle process is an operator on the bounded, continuous functionals on counting measures. We exhibit it here explicitly on a subclass of such functionals in its domain $\mathcal{D}_G$. We do so in terms of the characteristics of Equation 12.

5 Proposition. *Fix $l \geq 1$. The functions $f_1, \ldots, f_l$ belong to $\mathcal{C}_K^2$. The function g is a twice continuously differentiable function having compact support in $I\!R^l$. Let φ denote the functional $m \rightarrow g(mf_1, \ldots, mf_l)$ on M_b. The functional φ then belongs to $\mathcal{D}_G$ and $G\varphi$ satisfies:*

$$\begin{aligned} G\varphi(m) &= [\sum_{i=1}^{l} \beta(m; f_i)\partial_i g(y) + \frac{1}{2}\sum_{i=1}^{l}\sum_{j=1}^{l} q(m; f_i, f_j)\partial_{ij} g(y)]_{y=(mf_1,\ldots,mf_l)} \\ &\quad + \int_M \kappa(m; d\eta)(\varphi(m+\eta) - \varphi(m)). \end{aligned}$$

The symbols ∂_i and ∂_{ij} denote differentiation and partial differentiation with respect to the indicated coordinates.

This result is a consequence of the martingale dynamics of the particle proces and fruitful application of Itô's transformation formula for semimartingales. The details of the proof appear in Phelan (1993a).

5. A MARTINGALE PROBLEM

We define a martingale problem on the restriction of the generator of the particle process to the functionals in Proposition 4. The particle process clearly solves this problem. We show that it does so uniquely.

The Problem

The state space is the space $(M_b, \mathcal{M}_b)$ of bounded counting measures on E with its Borel sigma algebra relative to the vague topology. The space of sample paths is the Skorokhod space $\mathbb{D}(M_b)$ of vaguely right-continuous functions from $\mathbb{R}_+$ to M_b having vague limits from the left. The coordinate mappings on this space generate a filtration $\mathbf{F} = (\mathcal{F}_t), t \geq 0$ and the filtered space $(\mathbb{D}(M_b), \mathcal{F}, \mathbf{F})$. This is the canonical setting.

We recall the linear operator A of Equation 5. Let $\mathcal{D}_A$ denote its domain in the space of bounded, continuous functions on E. For each positive integer l, let $\mathbb{C}_K^\infty(\mathbb{R}^l)$ denote the space of infinitely, continuously differentiable functions with compact support in $\mathbb{R}^l$. We then introduce the set $\mathcal{D}_0$,

(19) $\mathcal{D}_0 = \{m \to g(mf_1, \ldots, mf_l) : f_1, \ldots, f_l \in \mathcal{D}_A, g \in \mathbb{C}_K^\infty(\mathbb{R}^l), l \geq 1\}$,

of bounded, continuous functionals that vanish at infinity on M_b. This then is a subset of the Banach space $C_0(M_b)$.

Proposition 4 implies that the domain $\mathcal{D}_G$ of the generator G contains $\mathcal{D}_0$. The restriction of G to the latter then defines a linear operator G_0, for example, and a subset $\{(\varphi, G\varphi) : \varphi \in \mathcal{D}_0\}$ in $C_0(M_b) \times C_0(M_b)$. We define our martingale problem on this set.

In particular, let P_0 be a probability measure on $(M_b, \mathcal{M}_b)$. And let P be a probability measure on $(\mathbb{D}(M_b), \mathcal{F})$. Finally, let X denote the canonical process on $(\mathbb{D}(M_b), \mathcal{F}, \mathbf{F}, P)$. We say that X is a solution process or that P is a solution measure to the martingale problem on the pair (G_0, P_0) whenever the law of X_0 is P_0 and the process $t \to X_t^\varphi$,

$$X_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t ds G_0 \varphi(X_s), \tag{20}$$

is a well-defined martingale for every φ in $\mathcal{D}_0$.

This then is a martingale problem of the kind in Ethier and Kurtz (1986). There is also one here of the kind in Jacod and Shiryaev (1987). One defines the latter with respect to the triplet of would-be characteristics at Equation 12.

Uniqueness

The particle process of Section 2 solves our martingale problem. We show here that it solves the problem uniquely among solution processes with sample paths in $\mathbb{D}(M_b)$. That the latter requirement is no restriction follows from the next lemma.

6 Lemma. *Any solution to the martingale problem on* (G_0, P_0) *has a modification with sample paths in* $\mathbb{D}(M_b)$.

The next propostion then states our main result. The martingale problem is well-posed, the particle process its unique solution. The details of the proof appear in Phelan (1993b).

7 Proposition. *Let* $t \to M_t$ *denote the particle process of Section 2. Let* P *denote the law of the particle process on the canonical setting* $(\mathbb{D}(M_b), \mathcal{F}, \mathbf{F})$ *and let* P_0 *denote the law of* M_0 *on* $(M_b, \mathcal{M}_b)$. *We suppose that the system parameter* (b, c, π, k) *satisfies the regularity conditions of Section 2 and that* k *is bounded on* E. *The probability measure* P *is then the unique solution to the martigale problem on* (G_0, P_0).

This is a useful result from a statistical point of view. It implies that there is an explicit representation for a Girsanov transformation between absolutely continuous particle systems, a most useful result for likelihood inference. In which case the estimation equations in Phelan (1992a) reduce nicely to the score function itself. Nevertheless, partial observational schemes such as in Phelan (1992b) still demand a more liberal class of estimation functions.

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